TILTING MUTATION OF WEAKLY SYMMETRIC ALGEBRAS AND STABLE EQUIVALENCE

ALEX DUGAS

ABSTRACT. We consider tilting mutations of a weakly symmetric algebra at a subset of simple modules, as recently introduced by T. Aihara. These mutations are defined as the endomorphism rings of certain tilting complexes of length 1. Starting from a weakly symmetric algebra A, presented by a quiver with relations, we give a detailed description of the quiver and relations of the algebra obtained by mutating at a single loopless vertex of the quiver of A. In this form the mutation procedure appears similar to, although significantly more complicated than, the mutation procedure of Derksen, Weyman and Zelevinsky for quivers with potentials. By definition, weakly symmetric algebras connected by a sequence of tilting mutations are derived equivalent, and hence stably equivalent. The second aim of this article is to describe explicitly the images of the simple modules under such a stable equivalence. As an application we answer a question of Asashiba on the derived Picard groups of a class of symmetric algebras of finite representation type. We conclude by introducing a mutation procedure for maximal systems of orthogonal bricks in a triangulated category, which is motivated by the effect that a tilting mutation has on the set of simple modules in the stable category.

Motivated by work of Okuyama and Rickard in modular representation theory, T. Aihara has recently introduced the notion of tilting mutation for symmetric algebras [1]. Roughly speaking, these tilting mutations are defined as endomorphism rings of special tilting complexes of length 1 that were first studied by Okuyama [15, 10] and have since proved quite useful in the construction of derived equivalences between blocks of finite groups as well as general symmetric algebras [4, 8, 9, 6]. These same tilting complexes have also been used by Vitoria [19] to establish derived equivalences between the Jacobian algebras of certain pairs of quivers with potential which are linked by a mutation in the sense of Derksen, Weyman and Zelevinsky [5]. Although these Jacobian algebras are often infinite-dimensional, this nice correspondence between the combinatorial mutation procedure of the quiver with potential and the homological mutation given by a derived equivalence warrants a further study of the combinatorics behind these derived equivalences in the finite-dimensional case.

The beginning of such a study is the first of two primary goals of the present article. We aim to give a combinatorial description of tilting mutation for weakly symmetric algebras. More precisely, given a weakly symmetric algebra A presented by a quiver with relations and a vertex i of the quiver at which there are no loops, we describe the quiver and relations of the endomorphism ring of a certain tilting complex associated to the vertex i. This endomorphism ring, which we denote $\mu_i^+(A)$, is also a weakly symmetric algebra, and we say that it is obtained by mutating A at the vertex i. After reviewing some general facts about tilting mutation in Section 1, we describe the quiver of the mutated algebra in Section 2, and the relations for the mutated algebra in Section 3, both in terms of the quiver and relations for A. We note that, in contrast to the mutation procedure for quivers with potential, in our setting the quiver for the mutated algebra typically depends on both the quiver and the relations of the original algebra. Nevertheless, there remain some unsurprising similarities between the effects of tilting mutations and quiver mutations on the quiver of our algebra. We see, for instance, reversal of arrows out of the vertex i as well as new arrows corresponding to paths of length two through i.

Secondly, we examine the effect of a tilting mutation inside the stable module category $\underline{\mathrm{mod}}$-A. Well-known work of Rickard has established that any two derived equivalent self-injective algebras are also stably equivalent [18]. However, little seems to be known about how the stable categories of two derived-equivalent self-injective algebras match up. Recently, Hu and Xi have given an explicit construction of a functor yielding such a stable equivalence that is induced by an equivalence of derived categories [11]. In Section 4, we use

2010 *Mathematics Subject Classification.* 16G10, 16E35, 18E30.

Key words and phrases. derived equivalence, stable equivalence, quiver mutation, weakly symmetric algebra.

this construction to compute the images in $\underline{\text{mod}}$-A of the simple modules over the mutated algebra, which are described in Theorem 4.1. As a consequence, we obtain a glimpse of how the stable categories of two often distinct weakly symmetric algebras, A and $\mu_i^+(A)$, may non-trivially identify with one another in a large class of examples. One potential application of this information involves the problem of lifting a stable equivalence to a derived equivalence. Namely, if one has an equivalence $\underline{\text{mod}}\text{-}B \to \underline{\text{mod}}\text{-}A$ of stable categories (of Morita type) which sends the simple B-modules to the images of the simple $\mu_i^+(A)$-modules, then a theorem of Linckelmann guarantees that B and $\mu_1^+(A)$ are Morita equivalent, and hence that B and A are derived equivalent [13]. In a similar vein, in Section 5 we apply our result to resolve a question of Asashiba concerning the derived Picard groups of a class of symmetric algebras of finite representation type [3]. In particular, we show that a certain auto-equivalence of $\underline{\text{mod}}$-A lifts to an auto-equivalence of the derived category $D^b(\text{mod-}A)$ by comparing the effects of these equivalences on the simple modules in $\underline{\text{mod}}$-A.

Finally, motivated by Theorem 4.1, in Section 6 we abstract the effect of a tilting mutation in $\underline{\text{mod}}$-A to a Hom-finite triangulated category $\mathcal{T}$ (with some additional hypotheses). The main idea is to view the images of the simple $\mu_i^+(A)$-modules in $\underline{\text{mod}}$-A as a mutation of the set of simple A-modules. We develop this notion of mutation inside a triangulated category $\mathcal{T}$ for *maximal systems of orthogonal bricks*, which are sets of objects which homologically resemble the set of simple modules inside a stable category. Our main result here is that the set of maximal systems of orthogonal bricks in $\mathcal{T}$ is closed under mutation. In particular, we obtain a way to keep track of successive tilting mutations to an algebra A inside the stable category of A by successively mutating the set of simple A-modules. Furthermore, such iterations of this mutation procedure will typically produce many nontrivial examples of maximal systems of orthogonal bricks in $\underline{\text{mod}}$-A, for a given algebra A.

Throughout this article, we typically work with right modules and write morphisms on the left, composing them from right to left. Likewise, paths in a quiver Q will be composed from right to left, and we often identify them with morphisms between projective right modules over (a quotient of) the path algebra. For an arrow α in a quiver, we shall write $s(\alpha)$ and $t(\alpha)$ for the source and target of α respectively. Moreover, if p and q are paths in Q we set

$$p/q = \begin{cases} p', & \text{if } p = p'q \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad q\backslash p = \begin{cases} p', & \text{if } p = qp' \\ 0, & \text{otherwise,} \end{cases}$$

and we extend these path-division operations linearly to k-linear combinations of paths p and p' in the obvious way. For an algebra A, we write $K(A)$ for the homotopy category of complexes of right A-modules. We use complexes with differential of degree 1 and define the degree-shifts of a complex $(X^\bullet, \delta)$ by $X[i]^p = X^{i+p}$ and $\delta[i] = (-1)^i\delta$ for $i \in \mathbb{Z}$. We occasionally signal the degree-0 term of a complex by underlining it, and we identify mod-A with complexes concentrated in degree 0. We also frequently make use of a shorthand for matrices of morphisms, writing merely $[f_{ij}]$ in square brackets instead of a full matrix to denote a morphism $f : \oplus_{i\in I} X_i \to \oplus_{j\in J} Y_j$ where $f_{ij} : X_i \to Y_j$ for each i, j. As is standard, we view elements of these direct sums as column vectors so that the morphism f corresponds to left "multiplication" by the matrix $[f_{ij}]$.

1. Tilting mutations

We assume that $A = k\Delta/I$ is a weakly symmetric k-algebra, presented as the path algebra of a quiver Δ modulo an admissible ideal I of relations. We let $J = k\Delta_{\geq 1}/I$ be the Jacobson radical of A, and we write $\Delta_0 = \{1, 2, \ldots, n\}$ and Δ_1 for the vertices and arrows of Δ respectively. For $U \subseteq \Delta_0$, we shall write $e_U = \sum_{i\in u} e_i$ for the sum of the corresponding primitive idempotents of A, and we shall write $P_U = e_U A$ for the corresponding projective A-module. We also write $Q_U = (1 - e_U)A$ so that $A_A \cong P_U \oplus Q_U$. Letting $f_U : P_U \to L_U$ be a minimal left add(Q_U)-approximation of P_U, it is not hard to see that

$$T_U = [\underline{P_U} \xrightarrow{f_U} L_U] \oplus Q_U[-1]$$

is a tilting complex concentrated in degrees 0 and 1. Similarly, if $g_U : R_U \to P_U$ is a minimal right add(Q_U)-approximation of P_U, then

$${}_UT = [R_U \xrightarrow{g_U} \underline{P_U}] \oplus Q_U[1]$$

is a tilting complex concentrated in degrees -1 and 0.

Definition 1.1. *Let $U \subseteq \Delta_0$. The* **right (tilting) mutation of** A **at** U *is the algebra*

$$\mu_U^+(A) = \mathrm{End}_{K(A)}(T_U),$$

and the **left (tilting) mutation of** A **at** U *is the algebra*

$$\mu_U^-(A) = \mathrm{End}_{K(A)}({}_UT).$$

Remark. Our notation differs somewhat from Aihara's in [1]. Namely, for a subset $U \subseteq \Delta_0$, Aihara defines a tilting complex $T(U)$, which is isomorphic to the complex ${}_{\bar{U}}T[1]$, where $\bar{U} = \Delta_0 \setminus U$. It follows that the tilting mutation of A at the vertex i, as defined by Aihara, coincides with the left tilting mutation $\mu_i^-(A)$ of A at i in the notation introduced above.

When we consider a sequence of such mutations, it is convenient to use the same indexing set for the vertices of the quivers of each mutated algebra. We employ the following convention: The vertices of the quiver of $\mu_U^+(A)$ correspond to the indecomposable summands of T_U. If $i \notin U$, then e_iA (as a complex concentrated in degree 1) is a summand of T_U and we keep the label i for the corresponding vertex of the new quiver. If $i \in U$, then we use i for the vertex corresponding to the summand $[\underline{P_i} \xrightarrow{f_i} L_{U,i}]$ of T_U, where f_i is a minimal left $\mathrm{add}(Q_U)$-approximation (later we will instead write i' for the vertex of the new quiver Δ' corresponding to $i \in \Delta_0$).

With these conventions we see that right and left mutations on the same subset of vertices yield inverse operations.

Lemma 1.2. *For any $U \subseteq \Delta_0$ we have*

$$\mu_U^-(\mu_U^+(A)) \cong A \cong \mu_U^+(\mu_U^-(A)).$$

Proof. Let $F : D^b(A) \to D^b(B)$ be the equivalence induced by T_U, where $B = \mu_U^+(A) \cong \mathrm{End}_{K(A)}(T_U)$. We write P'_U and Q'_U for the projective summands of B given by $F([\underline{P_U} \to L_U])$ and $F(Q_U[-1])$ respectively. The inverse equivalence then corresponds to the tilting complex $F(A)$ in $K(B)$. To calculate $F(A)$, observe that we have the triangle

$$[\underline{P_U} \to L_U] \longrightarrow A \xrightarrow{\begin{pmatrix} f_U & 0 \\ 0 & 1 \end{pmatrix}} L_U \oplus Q_U \longrightarrow .$$

Applying F yields a triangle in $K(B)$

$$P'_U \longrightarrow F(A) \longrightarrow F(L_U \oplus Q_U) \longrightarrow$$

with $F(L_U \oplus Q_U) \in \mathrm{add}(Q'_U[1])$ since $L_U \oplus Q_U \in \mathrm{add}(Q_U) \subset \mathrm{add}(T_U[1])$. It follows that $F(A)$ is concentrated in degrees 0 and -1, with P'_U in degree 0 and with its degree-(-1) term in $\mathrm{add}(Q'_U)$. Since we know that $F(A)$ is a tilting complex, it follows that the component $F(L_U)[-1] \to P'_U$ of the connecting morphism in the above triangle must be a right $\mathrm{add}(Q'_U)$-approximation of P'_U. Hence, $F(A)$ coincides with the tilting complex ${}_UT'$ constructed over B in the definition of $\mu_U^-(B)$. The second isomorphism is proved symmetrically. □

For the remainder of this article, we focus on the case where $U = \{1\}$ consists of a single vertex $1 \in \Delta_0$ at which there are no loops. To simplify our notation we shall henceforth replace the subscripts U used above by 1, or omit them altogether when there is no chance of confusion. Our first goal is to describe the quiver Δ' and relations I' of $B = \mu_1^+(A)$.

2. Arrows in the mutated quiver

We continue to assume that $A = k\Delta/I$ is a weakly symmetric k-algebra and $1 \in \Delta_0 = \{1, \ldots, n\}$ is a vertex at which there are no loops. We wish to describe the quiver Δ' of $B = \mu_1^+(A)$ in this case. As remarked already, the vertices of Δ' can be identified with the summands of T, and thus with the vertices of Δ. We shall write $\Delta'_0 = \{1', \ldots, n'\}$ where $1'$ corresponds to $[\underline{P_1} \to L]$ and i' corresponds to $P_i[-1]$ for each $i \neq 1$. Furthermore, notice that the former summand of T has the form $P_1 \xrightarrow{[\gamma]} \oplus_{s(\gamma)=1} P_{t(\gamma)}$. The rest of this section will focus on describing the arrows in Δ'. We will see that these may arise in several possible ways, depending on the arrows of Δ as well as on certain relations in I.

By our conventions, the arrows in Δ' from i' to j' correspond to (a k-basis of) the irreducible maps in $\mathrm{add}(T)$ between the corresponding summands of T. For simplicity, our initial description of the arrows in Δ' includes some maps which may turn out to be reducible, once the relations are taken into account.

(A1) Arrows $i' \to 1'$ for $i \neq 1$. The arrows $i' \to 1'$ in Δ' are in one-to-one correspondence with the arrows $1 \to i$ in Δ. Moreover, if $\alpha : 1 \to i$ in Δ, we denote by α^* the arrow corresponding to the irreducible map:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & P_i \arrow[d, "1_\alpha"] \\ P_1 \arrow[r, "{[\gamma]}"] & \bigoplus_{s(\gamma)=1} P_{t(\gamma)}. \end{tikzcd}$$

(A2) Arrows $1' \to i'$ for $i \neq 1$. Given an irreducible map

$$\begin{tikzcd} P_1 \arrow[r, "{[\gamma]}"] \arrow[d] & \bigoplus_{s(\gamma)=1} P_{t(\gamma)} \arrow[d, "{[f_\gamma]}"] \\ 0 \arrow[r] & P_i \end{tikzcd}$$

we must first have $\sum_\gamma f_\gamma \gamma = 0$ in A. That is, we must have a relation $r = \sum_\gamma f_\gamma \gamma \in I$, where the sum ranges over all arrows γ starting in 1 and f_γ is a linear combination of paths from $t(\gamma)$ to i. Since this map is assumed nonzero, we cannot have $r \in IJe_1$. Furthermore, for this map to be irreducible, the relation r must be left-minimal in the sense that it does not belong to $J(1-e_1)Ie_1$. For any such left-minimal relation r, we shall write r^* for the corresponding arrow $1' \to i'$ in Δ'. To obtain a complete set of such arrows r^* for Δ', we need only let r run through a k-basis of $(1-e_1)Ie_1/(1-e_1)(J(1-e_1)I + IJ)e_1 \cong (1-e_1)(I/(J(1-e_1)I + IJ))e_1$.

Note: When we speak of a basis of the quotient of an ideal in a path algebra kQ, such as I/I' where $I' \subseteq I \subseteq kQ$, we really mean a set $\mathcal{B}$ of linear combinations of paths in $I \subseteq kQ$ whose images in I/I' form a k-basis. Moreover, we always assume that each $p \in \mathcal{B}$ is a linear combination of paths with the same source and target vertices in Q.

(A3) Old arrows $i' \to j'$ for $i, j \neq 1$. If $\beta : i \to j$ is an arrow in Δ, then Δ' contains an arrow $\beta' : i' \to j'$ corresponding to the map $\beta[-1] : P_i[-1] \to P_j[-1]$ in $\mathrm{add}(T)$. However, in certain cases this map may fail to be irreducible. For example, in the relation $(r/\alpha)' - r^*\alpha^*$ introduced in (R2) of the next section, we would have $r/\alpha = \beta$ if $r := \beta\alpha$ is a relation in I.

(A4) New arrows $i' \to j'$ for $i, j \neq 1$. If $\beta : i \to 1$ and $\alpha : 1 \to j$ are arrows in Δ with $\alpha\beta \notin I$, then the corresponding map $\alpha\beta : e_iA[-1] \to e_jA[-1]$ may be irreducible in $\mathrm{add}(T)$. This yields an arrow $(\alpha\beta)' : i' \to j'$ in Δ'. In certain cases, this map may still be reducible: for instance, if I contains a relation of the form $\alpha\beta - p$ for some path p that does not pass through 1. Alternatively, it is possible that the various irreducible maps obtained in this way are not linearly independent: for instance, if I contains a relation of the form $\sum_i \alpha_i\beta_i$. All of these possibilities will be handled by the relations I' described in the next section. (Thus our initial description of I' is not necessarily an admissible ideal in the path algebra $k\Delta'$.)

(A5) Loops $1' \to 1'$. The quiver Δ' does not contain any loops at $1'$. Consider a radical morphism

$$\begin{tikzcd} P_1 \arrow[r, "{[\gamma]}"] \arrow[d, "f_0"] & \bigoplus_{s(\gamma)=1} P_{t(\gamma)} \arrow[d, "f_1"] \\ P_1 \arrow[r, "{[\gamma]}"] & \bigoplus_{s(\gamma)=1} P_{t(\gamma)} \end{tikzcd} .$$

Since Δ contains no loops at 1, f_0 factors through the left approximation $[\gamma]$. It follows that, up to homotopy, f_0 may be chosen to be 0. However, then the map $(0, f_1)$ will factor through $[\gamma^*] : \oplus_{s(\gamma)=1} P'_{t(\gamma)} \to P'_1$, and hence it is not irreducible.

3. Relations for the mutated algebra

We now describe the ideal I' of relations on the quiver Δ' so that $B = \mu^+(A) \cong k\Delta'/I'$. These relations can emerge in various ways, and we divide them up based on their starting and ending vertices. As mentioned already, the ideal I' may fail to be contained in $k\Delta'_{\geq 2}$. This means that some of the arrows in Δ' as described in the previous section may turn out to be redundant. Nevertheless, it is convenient to include them here, as they lead to more uniform descriptions of Δ' and I'.

To describe the relations in I', we will need a natural way of translating paths in Δ to paths in Δ'. Assume that p is a path from i to j in Δ with $i, j \neq 1$. We define the corresponding path p' in Δ' inductively by the following rules:

- If $p = \beta$ is an arrow, then set $p' := \beta'$ as in (A3) in the previous section.
- If $p = \alpha\beta$ for arrows $\beta : i \to 1$ and $\alpha : 1 \to j$ in Δ, then set $p' := (\alpha\beta)'$ as in (A4) in the previous section.
- If $p = p_1 p_2$ for a path p_1 that does not start at 1, then set $p' := p_1' p_2'$.

A simple induction on the length of p shows that p' is well-defined. Of course, we can extend this correspondence linearly to linear combinations of paths. A less formal way of viewing this translation is by interpreting $p \in k\Delta$ as a map $P_i \to P_j$, which corresponds to a map $p' : P_i[-1] \to P_j[-1]$ in $\mathrm{add}(T)$ and thus to a (not necessarily unique) linear combination of paths from i' to j' in $k\Delta'$. However, by specifying the correspondence on the level of the path algebras, we can sidestep this issue of non-uniqueness.

(R1) Old relations $i' \to j'$ for $i, j \neq 1$. If $\rho \in e_j I e_i$, then the corresponding linear combination of paths ρ' will be a relation from i' to j' in I'. Notice that we may obtain minimal relations for B' in this way, even starting from non-minimal relations for A. For instance, if $\rho : i \to 1$ is a minimal relation in I and $\alpha : 1 \to j$, then $(\alpha\rho)' \in I'$ could be minimal. In general, to obtain representatives of all the minimal relations of I' that arise in this way, we need to consider a k-basis for $(1 - e_1)[I/(J(1 - e_1)I + I(1 - e_1)J)](1 - e_1)$.

(R2) Suppose r is a (left minimal) relation in $e_i I e_1$ as in (A2) of the previous section. We can decompose $r = \sum_{s(\alpha)=1} (r/\alpha)\alpha$ where r/α is a linear combination of paths from $t(\alpha)$ to i. Then I' contains the relation

$$(r/\alpha)' - r^* \alpha^*$$

for each arrow $\alpha \in \Delta_1$ with source 1. The composite

$$\begin{tikzcd}
P_{t(\alpha)}[-1] \arrow[d, "\alpha^*"'] & 0 \arrow[r] \arrow[d] & P_{t(\alpha)} \arrow[d, "1_\alpha"] \\
T_1 \arrow[d, "r^*"'] & P_1 \arrow[r, "{[\gamma]}"] \arrow[d] & \bigoplus_{s(\gamma)=1} P_{t(\gamma)} \arrow[d, "{[r/\gamma]}"] \\
P_i[-1] & 0 \arrow[r] & P_i
\end{tikzcd}$$

clearly coincides with the map $(r/\alpha)' : P_{t(\alpha)}[-1] \to P_i[-1]$. Notice that $(r/\alpha)'$ may be a single arrow or even zero.

(R3) For any arrow $\beta : i \to 1$, we have the following relation

$$\sum_{s(\alpha)=1} \alpha^* (\alpha\beta)' : i' \to 1' \in I'.$$

To see this, notice that β provides a homotopy between this sum and the zero map.

$$\begin{tikzcd}
P_i[-1] \arrow[d, "{[(\alpha\beta)']}"'] & & 0 \arrow[r] \arrow[d] & P_i \arrow[d, "{[\alpha\beta]}"] \arrow[ddl, dashed, "\beta"'] \\
\bigoplus_{s(\alpha)=1} P_{t(\alpha)}[-1] \arrow[d, "{[\alpha^*]}"'] & & 0 \arrow[r] \arrow[d] & \bigoplus_{s(\alpha)=1} P_{t(\alpha)} \arrow[d, "1"] \\
T_1 & & P_1 \arrow[r, "{[\alpha]}"] & \bigoplus_{s(\alpha)=1} P_{t(\alpha)}
\end{tikzcd}$$

(R4) Suppose $\rho : 1 \to 1$ is a minimal relation in I, so that $\alpha\rho : 1 \to i$ is a relation inducing an arrow $(\alpha\rho)^*$ (as in (A2) of the previous section) for any arrow $\alpha : 1 \to i$. Then we have

$$\sum_{s(\alpha)=1} \alpha^*(\alpha\rho)^* : 1' \to 1' \in I'.$$

In fact the corresponding morphism $T_1 \to T_1$ is null-homotopic via the map $[\rho/\gamma]$ where $\rho = \sum_{s(\gamma)=1}(\rho/\gamma)\gamma$.

$$\begin{tikzcd}
T_1 \arrow[d, "{[(\alpha\rho)^*]}"'] & & P_1 \arrow[r, "{[\gamma]}"] \arrow[d] & \bigoplus_{s(\gamma)=1} P_{t(\gamma)} \arrow[d, "{[\alpha\rho/\gamma]}"] \arrow[ddl, "{[\rho/\gamma]}"'] \\
\bigoplus_{s(\alpha)=1} P_{t(\alpha)}[-1] \arrow[d, "{[\alpha^*]}"'] & & 0 \arrow[r] \arrow[d] & \bigoplus_{s(\alpha)=1} P_{t(\alpha)} \arrow[d, "1"] \\
T_1 & & P_1 \arrow[r, "{[\alpha]}"] & \bigoplus_{s(\alpha)=1} P_{t(\alpha)}
\end{tikzcd}$$

(R5) The relations from $1' \to i'$ with $i \neq 1$ are the most difficult to describe explicitly. We can identify them by way of the following lemma.

Lemma 3.1. *A linear combination ρ of paths from $1'$ to i' is contained in I' if and only if $\rho\alpha^* \in I'$ for all arrows $\alpha \in \Delta_1$ with source 1 (if and only if the map $P_{t(\alpha)} \to P_i$ that corresponds to $\rho\alpha^*$ is zero in A for all such α).*

Proof. The forward direction is clear. Thus assume that $\rho\alpha^* \in I'$ for all arrows $\alpha : 1 \to i$ in Δ. Since the arrows of the form α^* are the only arrows with target $1'$ in Δ', we see that $\rho J_B = 0$ and thus $\rho \in \operatorname{soc} P_1'$. As B is weakly symmetric, we must have $\operatorname{soc} P_1' \cong S_1'$, which forces $i = 1$ (a contradiction) or $\rho = 0$ in B. □

Remark. Let $\rho : 1' \to i'$ be a minimal relation arising as above. Observe that by using the relations from (R3), we may replace any $r^*\alpha^*$ occuring in ρ by $(r/\alpha)'$. We may thus choose a basis of $(1 - e_1')(I'/(J'I' + I'J'))e_1'$ consisting of linear combinations of paths not containing any arrow r^* other than as the initial arrow.

We now use the above results to describe (almost minimal) projective resolutions of the simple right B-modules. We will use these resolutions in the next section to compute the images of these simples under the stable equivalence induced by the tilting complex T. We shall apply the general description of a projective resolution of a simple module $S_i = e_i(kQ/I)$ over an algebra $\Lambda = kQ/I$, which begins

$$\bigoplus_{p \in e_i(I/(IJ+JI))} e_{s(p)}\Lambda \xrightarrow{[\alpha\backslash p]} \bigoplus_{\alpha \in Q_1,\ t(\alpha)=i} e_{s(\alpha)}\Lambda \xrightarrow{[\alpha]} e_i\Lambda \longrightarrow S_i \to 0,$$

where the first direct sum is indexed by a k-basis of $e_i(I/(IJ + JI))$. This resolution is actually minimal if I is an admissible ideal (i.e., if the presentation of Λ by quiver with relations is minimal), but that is not always the case for the quiver and relations (Q', I') described above.

Thus a projective resolution of S_1' begins

$$\bigoplus_{p\in e_1(I/(IJ+JI))e_1} P_1' \oplus \bigoplus_{t(\beta)=1} P_{s(\beta)}' \overset{\left([(\alpha p)^*]\ [(\alpha\beta)']\right)}{\longrightarrow} \bigoplus_{s(\alpha)=1} P_{t(\alpha)}' \xrightarrow{[\alpha^*]} P_1' \longrightarrow S_1' \to 0, \tag{3.1}$$

where again the first direct sum is indexed by a k-basis of the specified set. The first map between projectives is determined by the arrows of type (A1) from section 2, and the second map is determined by the minimal relations described in (R3) and (R4) above. Similarly, a projective resolution for S_i' with $i \neq 1$ begins

$$\begin{aligned} &\bigoplus_{\rho\in e_i' I' e_1'} P_1' \oplus \bigoplus_{q\in e_i I e_1}\left[\bigoplus_{s(\delta)=1} P_{t(\delta)}'\right] \oplus \bigoplus_{p\in e_i I(1-e_1)} P_{s(p)}' \qquad \overset{\varphi}{\longrightarrow} \\ &\bigoplus_{r\in e_i I e_1} P_1' \oplus \bigoplus_{j\xrightarrow{\gamma}1\xrightarrow{\alpha}i} P_{s(\gamma)}' \oplus \bigoplus_{t(\beta)=i} P_{s(\beta)}' \quad \overset{\left([r^*]\ [(\alpha\gamma)']\ [\beta']\right)}{\longrightarrow} \quad P_i' \longrightarrow S_i' \to 0, \end{aligned} \tag{3.2}$$

where the map φ is given by

$$\varphi = \begin{pmatrix} 0 & -[\delta_{rq}\delta^*] & 0 \\ [(\alpha\gamma)'\backslash\rho] & [(\alpha\gamma\backslash q/\delta)'] & [(\alpha\gamma\backslash p)'] \\ [\beta'\backslash\rho] & [(\beta\backslash q/\delta)'] & [(\beta\backslash p)'] \end{pmatrix}.$$

Furthermore, in the summations ρ runs through a k-basis of $e_i'(I'/(J'I'+I'J'))e_1'$ consisting of relations as in (R5), q and r run through the same k-basis of $e_i(I/(J(1-e_1)I+IJ))e_1$ as in (A2) of the last section, and p runs through a k-basis of $e_i[I/(J(1-e_1)I+I(1-e_1)J)](1-e_1)$ as in (R1) above. Observe that the 0 in the upper-left entry of φ is a consequence of the choice of basis described in the Remark following Lemma 3.1. In particular, for any ρ belonging to such a basis of $e_i'(I'/(J'I'+I'J'))e_1'$ and any r, we have $r^*\backslash\rho = 0$.

4. Images of the simple modules under stable equivalence

Now that we have described the quiver and relations of the mutated algebra $B = \mu_1^+(A)$, which is derived equivalent to the original algebra A, our goal in the remainder of this paper is to study the stable equivalence between A and B which is induced by the tilting complex T. While it is well known, thanks to the work of Rickard [18], that any pair of derived equivalent self-injective algebras are also stably equivalent, only recently has a description of the functor yielding the induced stable equivalence been spelled out in detail by Hu and Xi [11]. In the next two sections we shall use this description, along with Rickard's original description of the functor yielding the equivalence $D^b(\text{mod-}B) \to D^b(\text{mod-}A)$ [17], to compute the images of the simple B-modules in $\underline{\text{mod}}$-A. We shall write $S_1, \ldots, S_n$ and $S_1', \ldots, S_n'$ for the simple modules (up to isomorphism) over A and B respectively. We begin by summarizing the general process required to calculate the image of an arbitrary B-module X in $\underline{\text{mod}}$-A.

We let $F : D^-(\text{mod-}B) \to D^-(\text{mod-}A)$ be the equivalence constructed by Rickard in [17] that corresponds to the tilting complex $T \in K^b(\text{proj-}A)$, and we will denote by $\underline{F}$ the induced stable equivalence $\underline{\text{mod}}\text{-}B \to \underline{\text{mod}}\text{-}A$. Following Rickard, to construct $F(X)$ for $X \in \text{mod-}B$ one starts with a projective resolution $P^\bullet$ of X and then uses the equivalence proj-$B \approx \text{add}(T)$ to replace each term and each differential of this complex with the corresponding summands of $T \in K^b(\text{proj-}A)$ and the corresponding morphisms between them. The result is nearly a double complex over proj-A, but the square of the differential is zero in only one direction, while in the other direction it is zero only up to homotopy. Consequently, Rickard defines $F(X) \in K^-(\text{proj-}A)$ to be the total complex, but with differential given by a sum of maps of bi-degrees $(p, 1-p)$ for $p \geq 0$.

Since T is concentrated in degrees 0 and 1 the resulting complex $F(X)$ will be concentrated in degrees ≤ 1. Furthermore, by [11], $F(X)$ has homology concentrated in degrees 0 and 1 and is isomorphic in $K^-(\text{mod-}A)$ to a unique radical complex of the form

$$0 \to Q^0 \to Q^1 \to 0$$

where $Q^1 \in \text{add}(Q_1) = \text{add}((1-e_1)A)$. One then sets $\underline{F}(X) := Q^0$ in $\underline{\text{mod}}$-A.

Theorem 4.1. *With notation as above, we have $\underline{F}(S_1') \cong S_1$ and for each $i \neq 1$, $\underline{F}(S_i') \cong e_iJ(1-e_1)A$, which can also be described as the largest submodule of* $\mathrm{rad}P_i$ *without a factor of S_1 in its top Loewy layer.*

Proof. We first compute $\underline{F}(S_1')$. Following Rickard's construction of the functor F, we must first translate the projective resolution (3.1) into $K^b(\mathrm{add}(T))$.

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus_p \left[\bigoplus_{s(\gamma)=1} P_{t(\gamma)}\right] \oplus \bigoplus_{t(\beta)=1} P_{s(\beta)} \arrow[r, "{([\alpha p/\gamma]\ [\alpha\beta])}"] \arrow[drr, dashed, "{([p/\gamma]\ [\beta])}"'] & \oplus_{s(\alpha)=1} P_{t(\alpha)} \arrow[r, "1"] & \bigoplus_{s(\alpha)=1} P_{t(\alpha)} \arrow[r] & 0 \\
\cdots \arrow[r] & \bigoplus_p P_1 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & P_1 \arrow[r] \arrow[u, "{[\alpha]}"'] & 0
\end{tikzcd}$$

The associated total complex giving $F(S_1')$, incorporating the map $([p/\gamma]\ [\beta])$ of bi-degree $(2,-1)$, is now given by (degrees ≥ -1 are shown)

$$\cdots \to \bigoplus_p \left[\bigoplus_\gamma P_{t(\gamma)}\right] \oplus \bigoplus_{t(\beta)=1} P_{s(\beta)} \xrightarrow{\psi} P_1 \oplus \bigoplus_{s(\alpha)=1} P_{t(\alpha)} \xrightarrow{([\alpha]\ I)} \bigoplus_{s(\alpha)=1} P_{t(\alpha)} \to 0,$$

where ψ is given by

$$\psi = \begin{pmatrix} [p/\gamma] & [\beta] \\ -[\alpha p/\gamma] & -[\alpha\beta] \end{pmatrix}.$$

Finally, the above complex is homotopy equivalent to the complex

$$\cdots \to \bigoplus_p \left[\bigoplus_\gamma P_{t(\gamma)}\right] \oplus \bigoplus_{t(\beta)=1} P_{s(\beta)} \xrightarrow{([p/\gamma]\ [\beta])} P_1 \to 0,$$

concentrated in degrees ≤ 0. Thus, by [11] we have $\underline{F}(S_1') \cong \mathrm{coker}\ ([p/\gamma]\ [\beta]) \cong S_1$ since the second component $[\beta]$ of this map already maps onto rad P_1.

We now consider the image of the projective resolution (3.2) inside $K^b(\mathrm{add}(T))$.

$$\begin{tikzcd}
\cdots \arrow[r] & \begin{matrix} \bigoplus_\rho \left[\bigoplus_{s(\epsilon)=1} P_{t(\epsilon)}\right] \\ \bigoplus_q \left[\bigoplus_{s(\delta)=1} P_{t(\delta)}\right] \\ \bigoplus_p P_{s(p)} \end{matrix} \arrow[r, "\tilde{\varphi}"] & \begin{matrix} \bigoplus_r \left[\bigoplus_{s(\delta)=1} P_{t(\delta)}\right] \\ \bigoplus_{j \xrightarrow{\gamma} 1 \xrightarrow{\alpha} i} P_{s(\gamma)} \\ \bigoplus_{t(\beta)=i} P_{s(\beta)} \end{matrix} \arrow[r, "{([r/\delta]\ [\alpha\gamma]\ [\beta])}"] & P_i \arrow[r] & 0 \\
\cdots \arrow[r] & \bigoplus_\rho P_1 \arrow[r] \arrow[u] & \bigoplus_r P_1 \arrow[r] \arrow[u, "{\begin{pmatrix} \oplus_r[\delta] \\ 0 \\ 0 \end{pmatrix}}"'] & 0 \arrow[u] &
\end{tikzcd}$$

Here $\tilde{\varphi}$ is given by

$$\tilde{\varphi} = \begin{pmatrix} 0 & -I & 0 \\ [\alpha\gamma\backslash\hat{\rho}_\epsilon] & [\alpha\gamma\backslash q/\delta] & [\alpha\gamma\backslash p] \\ [\beta\backslash\hat{\rho}_\epsilon] & [\beta\backslash q/\delta] & [\beta\backslash p] \end{pmatrix}$$

where $\hat{\rho}_\epsilon$ is a linear combination of paths from $t(\epsilon)$ to i defined as follows. Start by writing $\rho : 1' \to i'$ as $\rho = \sum_r (\rho/r^*)r^*$, and note that for each arrow ϵ with source 1, r/ϵ is a linear combination of paths from $t(\epsilon)$ to i. Moreover the maps induced by $(r/\epsilon)'$ and $r^*\epsilon^*$ coincide (by relation (R2)). Note also that each ρ/r^* is a linear combination of paths in Δ' from some vertex $j' \neq 1'$ to i'. We may thus write $\rho/r^* = \rho_r'$ for a linear combination of paths $\rho_r : j \to i$ in Δ. We now set $\hat{\rho}_\epsilon = \sum_r \rho_r(r/\epsilon)$. As $\rho \in I'$, we must have each $\hat{\rho}_\epsilon \in I$ by Lemma 3.1.

We let $\mathcal{B}$ be a set of elements in $k\Delta$, each normed by a primitive idempotent on both sides, which yields a basis for $e_i(I/(J(1-e_1)I + I(1-e_1)J))$. (Note that we may assume that $\mathcal{B}e_1$ and $\mathcal{B}(1-e_1)$, respectively, are the bases through which r and p run in the diagram above, and in our continuing notation.)

Lemma 4.2. *In $k\Delta$, we have*

$$e_iI = k \cdot \mathcal{B} + \mathcal{B}(1-e_1)J + e_iJ(1-e_1)I = \mathcal{B} \cdot k\Delta + e_iJ(1-e_1)I.$$

In particular, we have

$$e_iI(1-e_1) = \mathcal{B}(1-e_1)k\Delta(1-e_1) + e_iJ(1-e_1)I(1-e_1).$$

Proof. Using $e_iI = k\mathcal{B} + e_iJ(1-e_1)I + e_iI(1-e_1)J$, induction on $m \geq 1$ shows that

$$e_iI \subseteq k\mathcal{B} + \mathcal{B}(1-e_1)J + e_iJ(1-e_1)I + e_iI[(1-e_1)J]^m.$$

Since the reverse inclusion clearly holds, we must have equality. The assumption that I is an admissible ideal of $k\Delta$ (i.e., that $J^N \subseteq I \subseteq J$ for some N) implies that by taking m sufficiently large we obtain $e_iI[(1-e_1)J]^m \subseteq e_iI(1-e_1)I \subseteq e_iJ(1-e_1)I$. This establishes the first equality. The second equality in the first line now follows since this last quantity clearly contains the middle expression, while being contained in e_iI. We get the identity of the second line by multiplying the first equality by $(1-e_1)$ on the right, and then repeating the argument for the second equality. $\square$

Corollary 4.3. *Each $\hat{\rho}_\epsilon$ can be written in the form*

$$\hat{\rho}_\epsilon = \sum_{p \in \mathcal{B}(1-e_1)} pa_p + x$$

where each $a_p \in k\Delta$ and $x \in e_iJ(1-e_1)I(1-e_1)$. As a consequence the map $\begin{pmatrix} (\alpha\gamma)\backslash\hat{\rho}_\epsilon \\ \beta\backslash\hat{\rho}_\epsilon \end{pmatrix} : P_{t(\epsilon)} \to P_{s(\gamma)} \oplus P_{s(\beta)}$ can be factored through $\begin{pmatrix} [\alpha\gamma\backslash p] \\ [\beta\backslash p] \end{pmatrix} : \oplus_p P_{s(p)} \to P_{s(\gamma)} \oplus P_{s(\beta)}$.

Proof. The stated decomposition of $\hat{\rho}_\epsilon$ follows from the fact that it is an element of $e_iI(1-e_1)$. Now notice that $(\alpha\gamma)\backslash x, \beta\backslash x \in I$. Thus $(\alpha\gamma)\backslash\hat{\rho}_\epsilon = \sum_p (\alpha\gamma)\backslash pa_p$ and $\beta\backslash\hat{\rho}_\epsilon = \sum_p \beta\backslash pa_p$, from which the second claim follows. $\square$

Lemma 4.4. *We have*

$$\underline{F}(S_i') \cong \operatorname{coker} \begin{pmatrix} [\alpha\gamma\backslash r] & [\alpha\gamma\backslash p] \\ [\beta\backslash r] & [\beta\backslash p] \end{pmatrix} : \bigoplus_r P_1 \oplus \bigoplus_p P_{s(p)} \to \bigoplus_{\gamma,\alpha} P_{s(\gamma)} \oplus \bigoplus_\beta P_{s(\beta)},$$

where p, r, γ, α and β range through the same index sets as above.

Proof. By definition of $\underline{F}$,

$$\underline{F}(S_i') \cong \operatorname{coker} \begin{pmatrix} \oplus_r[\delta] & 0 & -I & 0 \\ 0 & [(\alpha\gamma)\backslash\hat{\rho}_\epsilon] & [\alpha\gamma\backslash q/\delta] & [\alpha\gamma\backslash p] \\ 0 & [\beta\backslash\hat{\rho}_\epsilon] & [\beta\backslash q/\delta] & [\beta\backslash p] \end{pmatrix}.$$

Consider the following commutative exact diagram

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] & & \\
\bigoplus_q \left[\bigoplus_{s(\delta)=1} P_{t(\delta)}\right] \arrow[rr, "I"] \arrow[d, "\begin{pmatrix} 0 \\ 0 \\ I \\ 0 \end{pmatrix}"'] & & \bigoplus_r \left[\bigoplus_{s(\delta)=1} P_{t(\delta)}\right] \arrow[d, "\begin{pmatrix} -I \\ [(\alpha\gamma)\backslash r/\delta] \\ [\beta\backslash r/\delta] \end{pmatrix}"] & & \\
\begin{matrix} \bigoplus_r P_1 \\ \bigoplus_\rho \left[\bigoplus_{s(\epsilon)=1} P_{t(\epsilon)}\right] \\ \bigoplus_q \left[\bigoplus_{s(\delta)=1} P_{t(\delta)}\right] \\ \bigoplus_p P_{s(p)} \end{matrix} \arrow[rr, "\begin{pmatrix} \oplus_r[\delta] & 0 & -I & 0 \\ 0 & [(\alpha\gamma)\backslash\hat{\rho}_\epsilon] & [\alpha\gamma\backslash q/\delta] & [\alpha\gamma\backslash p] \\ 0 & [\beta\backslash\hat{\rho}_\epsilon] & [\beta\backslash q/\delta] & [\beta\backslash p] \end{pmatrix}"] \arrow[d, "\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}"'] & & \begin{matrix} \bigoplus_r \left[\bigoplus_{s(\delta)=1} P_{t(\delta)}\right] \\ \bigoplus_{j \xrightarrow{\gamma} 1 \xrightarrow{\alpha} i} P_{s(\gamma)} \\ \bigoplus_{t(\beta)=i} P_{s(\beta)} \end{matrix} \arrow[r] \arrow[d, "\begin{pmatrix} [\alpha\gamma\backslash r/\delta] & I & 0 \\ [\beta\backslash r/\delta] & 0 & I \end{pmatrix}"] & X \arrow[r] \arrow[d, "\cong"] & 0 \\
\begin{matrix} \bigoplus_r P_1 \\ \bigoplus_\rho \left[\bigoplus_{s(\epsilon)=1} P_{t(\epsilon)}\right] \\ \bigoplus_p P_{s(p)} \end{matrix} \arrow[rr, "\begin{pmatrix} [(\alpha\gamma)\backslash r] & [(\alpha\gamma)\backslash\hat{\rho}_\epsilon] & [(\alpha\gamma)\backslash p] \\ [\beta\backslash r] & [\beta\backslash\hat{\rho}_\epsilon] & [\beta\backslash p] \end{pmatrix}"] \arrow[d] & & \begin{matrix} \bigoplus_{j \xrightarrow{\gamma} 1 \xrightarrow{\alpha} i} P_{s(\gamma)} \\ \bigoplus_{t(\beta)=i} P_{s(\beta)} \end{matrix} \arrow[r] \arrow[d] & Y \arrow[r] & 0 \\
0 & & 0 & &
\end{tikzcd}$$

Combining the induced isomorphism between the cokernels in the above diagram with Lemma 4.2, we now have

$$\begin{aligned} \underline{F}(S'_i) &\cong \operatorname{coker} \begin{pmatrix} [(\alpha\gamma)\backslash r] & [(\alpha\gamma)\backslash\hat{\rho}_\epsilon] & [(\alpha\gamma)\backslash p] \\ [\beta\backslash r] & [\beta\backslash\hat{\rho}_\epsilon] & [\beta\backslash p] \end{pmatrix} \\ &\cong \operatorname{coker} \begin{pmatrix} [\alpha\gamma\backslash r] & [\alpha\gamma\backslash p] \\ [\beta\backslash r] & [\beta\backslash p] \end{pmatrix}. \quad \square \end{aligned}$$

Lemma 4.5. *We have*

$$\begin{aligned} \underline{F}(S'_i) &\cong \operatorname{im}([\alpha\gamma] \ [\beta]) : \bigoplus_{j \xrightarrow{\gamma} 1 \xrightarrow{\alpha} i} P_{s(\gamma)} \oplus \bigoplus_{t(\beta)=i} P_{s(\beta)} \to P_i \\ &= e_i J(1-e_1)A. \end{aligned}$$

Proof. The claim follows from the exactness of the following sequence.

$$\begin{matrix} \bigoplus_r P_1 \\ \bigoplus_p P_{s(p)} \end{matrix} \xrightarrow{\begin{pmatrix} [\alpha\gamma\backslash r] & [\alpha\gamma\backslash p] \\ [\beta\backslash r] & [\beta\backslash p] \end{pmatrix}} \begin{matrix} \bigoplus_{j \xrightarrow{\gamma} 1 \xrightarrow{\alpha} i} P_{s(\gamma)} \\ \bigoplus_{t(\beta)=i} P_{s(\beta)} \end{matrix} \xrightarrow{([\alpha\gamma] \ [\beta])} P_i$$

Thus suppose that $q = (q_{(\alpha,\gamma)}, q_\beta)_{(\alpha,\gamma),\beta} \in \ker\,([\alpha\gamma] \ [\beta])$. Then

$$\sum_{\alpha,\gamma} \alpha\gamma q_{(\alpha,\gamma)} + \sum_\beta \beta q_\beta \in e_i I = \mathcal{B} \cdot k\Delta + e_i J(1-e_1)I$$

by Lemma 4.2. Thus we can write

$$\sum_{\alpha,\gamma} \alpha\gamma q_{(\alpha,\gamma)} + \sum_\beta \beta q_\beta = \sum_{r \in \mathcal{B}e_1} r a_r + \sum_{p \in \mathcal{B}(1-e_1)} p a_p + x$$

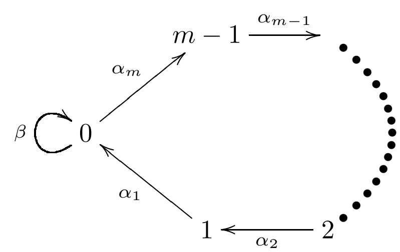
for $a_r, a_p \in k\Delta$ and $x \in e_i J(1-e_1)I$. Since $(\alpha\gamma)\backslash x, \beta\backslash x \in I$ for all α, γ and β, we have $q_{(\alpha,\gamma)} = \sum_r (\alpha\gamma)\backslash r a_r + \sum_p (\alpha\gamma)\backslash p a_p$ in $A = k\Delta/I$. Similarly, we have $q_\beta = \sum_r \beta\backslash r a_r + \sum_p \beta\backslash p a_p$ in A. As a result, we see

$$q = \begin{pmatrix} [\alpha\gamma\backslash r] & [\alpha\gamma\backslash p] \\ [\beta\backslash r] & [\beta\backslash p] \end{pmatrix} \begin{pmatrix} [a_r] \\ [a_p] \end{pmatrix}$$

in A. Thus q is in the image of the previous map, and the sequence is exact. □

5. Examples and applications

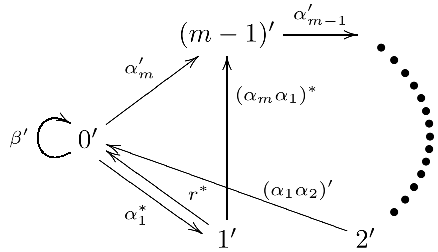
For our first example, we consider a standard self-injective algebra of finite representation type which is of type $(\mathbb{D}_{3m}, 1/3, 1)$ for $m \geq 2$. In fact, by Asashiba's classification theorem [2], any such algebra is derived equivalent to the algebra A presented by the quiver



and relations (i) $\alpha_1 \cdots \alpha_m = \beta^2$; (ii) $\overbrace{\alpha_i \cdots \alpha_m \beta \alpha_1 \cdots \alpha_i}^{m+2} = 0$ for all $i \in \{1, \ldots, m\} = \mathbb{Z}/\langle m \rangle$; and (iii) $\alpha_m \alpha_1 = 0$ [3].

Our interest in this specific example stems from the following problem raised by Asashiba in [3], where it is shown that nearly all auto-equivalences of the stable category of a standard self-injective algebra of finite type can be lifted to derived equivalences. Asashiba shows that there is essentially one stable auto-equivalence for the algebras of type $(\mathbb{D}_{3m}, s/3, 1)$ with $(3, s) = 1$ for which this problem is left unresolved. More specifically, the stable AR-quiver of such an algebra Λ is isomorphic to $\mathbb{Z}\mathbb{D}_{3m}/\langle \tau^{(2m-1)s} \rangle$, and the stable module category $\underline{\text{mod}}$-Λ is (the k-variety generated by) the mesh category $k(\mathbb{Z}\mathbb{D}_{3m}/\langle \tau^{(2m-1)s} \rangle)$ since Λ is standard. Hence, corresponding to the order-2 automorphism of $\mathbb{D}_{3m}$, we have an automorphism H of the AR-quiver of Λ and therefore of the category $\underline{\text{mod}}$-Λ as well, which fixes most indecomposables, but swaps each pair of indecomposables corresponding to the leafs of a sectional $\mathbb{D}_{3m}$ subquiver. By Asashiba's description of the stable picard group of Λ (i.e., the group of all auto-equivalences of $\underline{\text{mod}}$-Λ, modulo automorphisms), to show that this auto-equivalence is induced by an auto-equivalence of $D^b(\text{mod-}\Lambda)$ it suffices to show that at least one stable equivalence which does not induce a power of τ on the stable AR-quiver can be lifted to a derived equivalence. At least for the algebra A given above ($s = 1$), we will show that such a stable equivalence is induced by a tilting mutation, and hence by a derived equivalence. It should be possible to use covering theory to extend this result to the $\mathbb{Z}/s$-Galois covering of A which gives a representative algebra of type $(\mathbb{D}_{3m}, s/3, 1)$, but we do not pursue this here.

For the algebra A, we now see what happens when we perform a tilting mutation at the vertex 1. The quiver of the resulting endomorphism algebra contains arrows β' and α_i' for $3 \leq i \leq m$. We also have new arrows $(\alpha_1\alpha_2)'$, α_1^*, $(\alpha_m\alpha_1)^*$ and r^* where r can be taken to be $\beta^2\alpha_1$.



For the relations, we first consider those induced by the original relations of A. From (i) we get

$$(\alpha_1\alpha_2)'\alpha_3' \cdots \alpha_m' = (\beta')^2;$$

from (ii) we get

$$\alpha_i' \cdots \beta'(\alpha_1\alpha_2)' \cdots \alpha_i' = 0 \text{ for } 3 \leq i \leq m$$

as well as

$$(\alpha_1\alpha_2)' \cdots \alpha_m'\beta'(\alpha_1\alpha_2)' = 0$$

which is induced by the corresponding non-minimal relation from 2 to 0. From (R2) applied to the relations r and $\alpha_m\alpha_1$ we obtain

$$\begin{array}{rcl} (\beta')^2 & = & r^*\alpha_1^* \\ \alpha_m' & = & (\alpha_m\alpha_1)^*\alpha_1^*. \end{array}$$

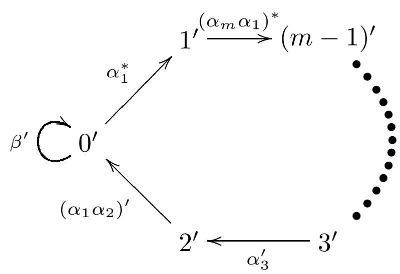
From (R3) we obtain

$$\alpha_1^*(\alpha_1\alpha_2)' = 0.$$

We do not obtain any relations from (R4), but (R5) yields the relation

$$r^* = (\alpha_1\alpha_2)' \cdots \alpha_{m-1}'(\alpha_m\alpha_1)^*$$

since we have $r^*\alpha_1^* = (\beta')^2 = (\alpha_1\alpha_2)' \cdots \alpha_m' = (\alpha_1\alpha_2)' \cdots \alpha_{m-1}'(\alpha_m\alpha_1)^*\alpha_1^*$. Thus the mutated algebra B can be presented by the quiver

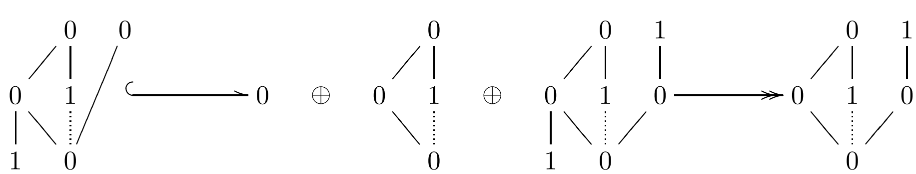


which is isomorphic to the quiver of A. Moreover, the relations above correspond precisely to the relations of A under such an isomorphism of quivers. Thus we have $\mu_1^+(A) \cong A$ in this example.

In light of this isomorphism, we may consider the induced derived equivalence F as an autoequivalence of $D^b(A)$, which induces the autoequivalence $\underline{F}$ of $\underline{\mathrm{mod}}$-A. Since this isomorphism identifies $0'$ with 0, $1'$ with $m-1$ and i' with $i-1$ for $2 \leq i \leq m-1$, Theorem 4.1 shows that $\underline{F}$ has the following effect on the simple A-modules

$$\begin{array}{rcl} \underline{F}(S_{m-1}) & \cong & S_1 \cong \Omega^3(S_{m-1}), \\ \underline{F}(S_0) & \cong & e_0J(1-e_1)A \cong \Omega^3(e_0A/\beta\alpha_1A), \\ \underline{F}(S_i) & \cong & e_{i+1}J(1-e_1)A \cong \Omega(S_{i+1}) \cong \Omega^3(S_i) \text{ for } 1 \leq i \leq m-2. \end{array}$$

We claim that no power of the syzygy functor Ω (or, equivalently, of τ) can have the same effect on the simples in $\underline{\mathrm{mod}}$-A. This is due to the fact that Ω coincides with τ^m on objects, while S_0 and $e_0A/\beta\alpha_1A$ are not in the same τ-orbit. The latter assertion can be seen by the fact that S_0 and $e_0A/\beta\alpha_1A$ occur as summands (shown below as the first two) of the middle term in the same almost split sequence:



Since the AR-quiver of A is isomorphic to $\mathbb{Z}\mathbb{D}_{3m}/\langle\tau^{2m-1}\rangle$, the indecomposable summands of the middle term of any almost split sequence must lie in distinct τ-orbits.

Next we consider the algebra A given by the quiver

$$\begin{tikzcd} & 1 \arrow[dl, shift left, "\beta_3"] \arrow[dr, shift left, "\alpha_1"] & \\ 3 \arrow[ur, shift left, "\alpha_3"] \arrow[rr, shift left, "\beta_2"] & & 2 \arrow[ul, shift left, "\beta_1"] \arrow[ll, shift left, "\alpha_2"] \end{tikzcd}$$

and the relations $\beta_i\alpha_i = 0 = \alpha_i\beta_i$ and $\alpha_{i+2}\alpha_{i+1}\alpha_i = \beta_i\beta_{i+1}\beta_{i+2}$ for each i (mod 3). Performing a tilting mutation at the vertex 1, we obtain arrows as follows:

(A1) Corresponding to the arrows α_1 and β_3, we have $\alpha_1^* : 2' \to 1'$ and $\beta_3^* : 3' \to 1'$.

(A2) Corresponding to the relations $\beta_3\beta_1\alpha_1 = 0$, $\alpha_1\alpha_3\beta_3 = 0$, $\alpha_1\alpha_3\alpha_2\alpha_1 = 0$ and $\beta_3\beta_1\beta_2\beta_3 = 0$ we obtain arrows $(\beta_3\beta_1\alpha_1)^* : 1' \to 3'$; $(\alpha_1\alpha_3\beta_3)^* : 1' \to 2'$; $(\alpha_1\alpha_3\alpha_2\alpha_1)^* : 1' \to 2'$ and $(\beta_3\beta_1\beta_2\beta_3)^* : 1' \to 3'$.

(A3) Corresponding to the arrows α_2 and β_2, we have $\alpha_2' : 2' \to 3'$ and $\beta_2' : 3' \to 2'$.

(A4) Corresponding to the nonzero paths of length 2 passing through 1, we have the arrows $(\alpha_1\alpha_3)' : 3' \to 2'$ and $(\beta_3\beta_1)' : 2' \to 3'$.

Following Section 3, we work out the relations to be:

(R1) From the relations in A between vertices 2 and 3, we obtain $\beta_2'\alpha_2' = \alpha_2'\beta_2' = 0$, $(\alpha_1\alpha_3)'\alpha_2' = \beta_2'(\beta_3\beta_1)'$ and $\alpha_2'(\alpha_1\alpha_3)' = (\beta_3\beta_1)'\beta_2'$.

(R2) The relations of the form $(r/\alpha)' = r^*\alpha^*$ where r runs through the relations used in (A2) above are $(\beta_3\beta_1)' = (\beta_3\beta_1\alpha_1)^*\alpha_1^*, 0 = (\beta_3\beta_1\alpha_1)^*\beta_3^*; (\alpha_1\alpha_3)' = (\alpha_1\alpha_3\beta_3)^*\beta_3^*, 0 = (\alpha_1\alpha_3\beta_3)^*\alpha_1^*; (\alpha_1\alpha_3)'\alpha_2' = (\alpha_1\alpha_3\alpha_2\alpha_1)^*\alpha_1^*; 0 = (\alpha_1\alpha_3\alpha_2\alpha_1)^*\beta_3^*; (\beta_3\beta_1)'\beta_2' = (\beta_3\beta_1\beta_2\beta_3)^*\beta_3^*; 0 = (\beta_3\beta_1\beta_2\beta_3)^*\alpha_1^*$. In particular, we see that the arrows $(\beta_3\beta_1)'$ and $(\alpha_1\alpha_3)'$ may be eliminated from the quiver.

(R3) For $\beta = \alpha_3, \beta_1$, we get the relations $\alpha_1^*(\alpha_1\alpha_3)' = 0$ and $\beta_3^*(\beta_3\beta_1)' = 0$ respectively.

(R4) Taking $\rho = \alpha_3\beta_3$ and $\rho = \beta_1\alpha_1$ yields $\alpha_1^*(\alpha_1\alpha_3\beta_3)^* = 0$ and $\beta_3^*(\beta_3\beta_1\alpha_1)^* = 0$ respectively. Taking $\rho = \alpha_3\alpha_2\alpha_1 - \beta_1\beta_2\beta_3$ yields $\alpha_1^*(\alpha_1\alpha_3\alpha_2\alpha_1)^* - \beta_3^*(\beta_3\beta_1\beta_2\beta_3)^* = 0$.

(R5) Combining relations from (R2), (R1) and (R2) again, we have $(\alpha_1\alpha_3\alpha_2\alpha_1)^*\alpha_1^* = (\alpha_1\alpha_3)'\alpha_2' = \beta_2'(\beta_3\beta_1)' = \beta_2'(\beta_3\beta_1\alpha_1)^*\alpha_1^*$ and $(\alpha_1\alpha_3\alpha_2\alpha_1)^*\beta_3^* = 0 = \beta_2'(\beta_3\beta_1\alpha_1)^*\beta_3^*$ by (R2). Thus we have $(\alpha_1\alpha_3\alpha_2\alpha_1)^* = \beta_2'(\beta_3\beta_1\alpha_1)^*$ using Lemma 3.1. Similarly, one checks that $(\beta_3\beta_1\beta_2\beta_3)^* = \alpha_2'(\alpha_1\alpha_3\beta_3)^*$.

Thus, after eliminating the redundant arrows, the quiver of the mutated algebra becomes

$$\begin{tikzcd} & 1' \arrow[dl, shift left, "(\beta_3\beta_1\alpha_1)^*"] \arrow[dr, shift left, "(\alpha_1\alpha_3\beta_3)^*"] & \\ 3' \arrow[ur, shift left, "\beta_3^*"] \arrow[rr, shift left, "\beta_2'"] & & 2' \arrow[ul, shift left, "\alpha_1^*"] \arrow[ll, shift left, "\alpha_2'"] \end{tikzcd}$$

and the relations show that this algebra is in fact isomorphic to the original algebra A. By Theorem 4.1, we have

$$\underline{F}(S_1') \cong S_1; \quad \underline{F}(S_2') \cong \begin{array}{ccc} 3 & & 3 \\ | & & / \\ 1 & / & \\ | & & \\ 2 & & \end{array}; \quad \underline{F}(S_3') \cong \begin{array}{ccc} 2 & & 2 \\ | & & / \\ 1 & / & \\ | & & \\ 3 & & \end{array}.$$

6. Mutations of maximal systems of orthogonal bricks

In Sections 2 and 3 we have described the quiver and relations for the algebra obtained via a tilting mutation at a single vertex without a loop. Even in this restricted setting, it would be interesting to understand the changes to the quiver and relations resulting from performing successive tilting mutations. While this problem may be too general to admit a nice answer, we can still apply the results of Section 4 to keep track of these successive mutations internally in $\underline{\text{mod}}$-A by following the images of the simple modules. In fact, this leads us to a way of mutating sets of objects that behave like simple modules in the stable category $\underline{\text{mod}}$-A.

For starters, suppose that A is a weakly symmetric algebra with simples $S_1, \dots, S_n$ such that $\mathrm{Ext}^1_A(S_1, S_1) = 0$, and let $B = \mu_1^+(A)$. If $\underline{F} : \underline{\mathrm{mod}}\text{-}B \to \underline{\mathrm{mod}}\text{-}A$ is the induced stable equivalence, Theorem 4.1 shows that the images of the simple B-modules under $\underline{F}$ are the modules S_i' defined by $S_1' = S_1$ and for $i \neq 1$ by the short exact sequences

$$0 \to S_i' \longrightarrow \Omega S_i \xrightarrow{f_i} S_1^{n_i} \to 0$$

where f_i is a minimal left $\mathrm{add}(S_1)$-approximation. In fact, we can replace these short exact sequence by distinguished triangles $S_i' \longrightarrow S_i[-1] \xrightarrow{f_i} S_1^{n_i} \to$ in the stable category $\underline{\mathrm{mod}}\text{-}A$, where $-[d]$ denotes the d^{th} power of the suspension functor Ω^{-1}. This observation motivates us to define a general mutation procedure on sets of objects resembling simples in certain triangulated categories.

We thus let $\mathcal{T}$ be a Hom-finite triangulated Krull-Schmidt k-category with suspension denoted by $-[1]$. In order for $\mathcal{T}$ to resemble the stable module category of a weakly symmetric algebra, we assume additionally that

$$\mathcal{T}(X, Y) \cong \mathcal{T}(Y, X[-1])$$

for all $X, Y \in \mathcal{T}$. These isomorphisms are not required to be natural, and as these Hom-spaces are only k-vector spaces, this is equivalent to asking that these two Hom-spaces always have equal dimensions. Such an equality holds, for instance, if $\mathcal{T}$ is -1-Calabi-Yau. In fact, $\underline{\mathrm{mod}}\text{-}A$ is -1-Calabi-Yau precisely when A is symmetric (and not semi-simple). Following Pogorzały [16], we say that a (finite) set $\mathcal{S} = \{S_1, \dots, S_n\}$ of objects of $\mathcal{T}$ is a *maximal system of orthogonal bricks* if the following hold:

(1) $\mathcal{T}(S_i, S_i) \cong k$ for all i.
(2) $\mathcal{T}(S_i, S_j) = 0$ for all $i \neq j$.
(3) For every nonzero $X \in \mathcal{T}$ there exists a j such that $\mathcal{T}(X, S_j) \neq 0$.
(4) $S_i[2] \not\cong S_i$ for all i.

Note that condition (3) is equivalent to (3') For every $X \in \mathcal{T}$ there exists a j such that $\mathcal{T}(S_j, X) \neq 0$ in light of the identity $\dim_k \mathcal{T}(X, Y) = \dim_k \mathcal{T}(Y, X[-1])$. We have also substituted condition (4) for the requirement (from Pogorzały's original definition) that $\tau S_i \not\cong S_i$ for all i, as these two are equivalent for the set of simples in the stable category of a weakly symmetric algebra.

Definition 6.1. *Suppose $\mathcal{S} = \{S_1, \dots, S_n\}$ is maximal system of orthogonal bricks in $\mathcal{T}$ such that $\mathcal{T}(S_i, S_i[1]) = 0$. We define the* **(right) mutation of $\mathcal{S}$ at** S_i *to be $\mu_i^+(\mathcal{S}) := \{S_1', \dots, S_n'\}$ where $S_i' = S_i$ and for $j \neq i$, S_j' is defined via a distinguished triangle*

$$S_j' \to S_j[-1] \xrightarrow{f_j} S_i^{n_j} \to$$

where f_j is a minimal left $\mathrm{add}(S_i)$*-approximation.*

Observe that in the above definition, since $\mathcal{T}(S_i, S_i) \cong k$, the map f_j has the form $(g_1, \dots, g_{n_j})^T : S_j[-1] \to S_i^{n_j}$ where $\{g_1, \dots, g_{n_j}\}$ is a k-basis for $\mathcal{T}(S_j[-1], S_i)$.

We first show that (right) mutation preserves the defining properties of maximal systems of orthogonal bricks.

Theorem 6.2. *Let $\mathcal{S} = \{S_1, \dots, S_n\}$ be a maximal system of orthogonal bricks in $\mathcal{T}$, and suppose that $\mathcal{T}(S_i, S_i[1]) = 0$. Then the set $\mu_i^+(\mathcal{S})$ is again a maximal system of orthogonal bricks in $\mathcal{T}$.*

Proof. Applying $\mathcal{T}(-, S_i)$ to the triangle $S_j' \to S_j[-1] \xrightarrow{f_j} S_i^{n_j} \to$ yields an exact sequence

$$\mathcal{T}(S_i^{n_j}, S_i) \to \mathcal{T}(S_j[-1], S_i) \xrightarrow{0} \mathcal{T}(S_j', S_i) \to \mathcal{T}(S_i^{n_j}[-1], S_i),$$

where the first map is an epimorphism by definition of f_j and the last term is 0 since S_i is assumed to have no self-extensions. Thus $\mathcal{T}(S_j', S_i) = 0$. Similarly applying $\mathcal{T}(S_i, -)$ to the same triangle yields the exact

sequence.

$$\begin{tikzcd}
\mathcal{T}(S_i, S'_j[-1]) \arrow[r] \arrow[d, "\cong"] & \mathcal{T}(S_i, S_j[-2]) \arrow[r] \arrow[d, "\cong"] & \mathcal{T}(S_i, S_i^{n_j}[-1]) \arrow[r] \arrow[d, "\cong"] & \mathcal{T}(S_i, S'_j) \arrow[r] & \mathcal{T}(S_i, S_j[-1]) \arrow[d, "\cong"] \\
\mathcal{T}(S'_j, S_i) \arrow[d, equal] & \mathcal{T}(S_j[-1], S_i) \arrow[d, equal] & \mathcal{T}(S_i^{n_j}, S_i) \arrow[d, equal] & & \mathcal{T}(S_j, S_i) \arrow[d, equal] \\
0 & k^{n_j} & k^{n_j} & & 0
\end{tikzcd}$$

We thus see that $\mathcal{T}(S_i, S'_j) = 0$. Now suppose that $j, l \neq i$, and consider the exact diagram obtained by applying $\mathcal{T}(S'_j, -)$ and $\mathcal{T}(-, S_l[-1])$ to the triangles defining S'_l and S'_j respectively.

$$\begin{tikzcd}
 & & \mathcal{T}(S_i^{n_j}[-1], S_l[-1]) & \\
\mathcal{T}(S'_j, S_i^{n_l}[-1]) \arrow[r, "0"] & \mathcal{T}(S'_j, S'_l) \arrow[r] & \mathcal{T}(S'_j, S_l[-1]) \arrow[r, "0"] \arrow[u, "0"] & \mathcal{T}(S'_j, S_i^{n_l}) \\
 & & \mathcal{T}(S_j[-1], S_l[-1]) \arrow[u] & \\
 & & \mathcal{T}(S_i^{n_j}, S_l[-1]) \arrow[u, "0"] &
\end{tikzcd}$$

To get the zero maps, we have used that $\mathcal{T}(S'_j, S_i^{n_l}[-1]) \cong \mathcal{T}(S_i^{n_l}, S'_j) = 0$, $\mathcal{T}(S_i, S_l) = 0$, $\mathcal{T}(S'_j, S_i) = 0$ and $\mathcal{T}(S_i^{n_j}, S_l[-1]) \cong \mathcal{T}(S_l, S_i^{n_j}) = 0$. Thus the remaining nonzero maps now yield isomorphisms

$$\mathcal{T}(S'_j, S'_l) \cong \mathcal{T}(S'_j, S_l[-1]) \cong \mathcal{T}(S_j[-1], S_l[-1]) \cong \mathcal{T}(S_j, S_l) \cong \begin{cases} k & j = l \\ 0 & j \neq l. \end{cases}$$

Next we check that condition (3) holds for $\mu_i^+(\mathcal{S})$. Assume that $\mathcal{T}(X, S'_j) = 0$ for all j for some nonzero $X \in \mathcal{T}$. In particular, $\mathcal{T}(X, S_i) = 0$ since $S'_i = S_i$. Thus for all $j \neq i$, applying $\mathcal{T}(X, -)$ to the defining triangle for S'_j shows that $\mathcal{T}(X, S_j[-1]) = 0$. Consequently, $\mathcal{T}(S_j, X) = 0$ for all $j \neq i$, which forces $\mathcal{T}(S_i, X) \neq 0$ by the dual of condition (3) for the maximal system of orthogonal bricks $\mathcal{S}$. Now let $g : S_i^r \to X$ be a minimal right $\operatorname{add}(S_i)$-approximation of X, where $r = \dim_k \mathcal{T}(S_i, X) > 0$, and define Y as the cone of g in a distinguished triangle

$$S_i^r \xrightarrow{g} X \longrightarrow Y \to .$$

Since $\mathcal{T}(S_i[1], S'_j) \cong \mathcal{T}(S'_j, S_i) = 0$ for all $j \neq i$, we see that $\mathcal{T}(Y, S'_j) = 0$ for all $j \neq i$. Next, we apply $\mathcal{T}(S_i, -)$ to the above triangle to get

$$\begin{tikzcd}
\mathcal{T}(S_i, X[-1]) \arrow[r] \arrow[d, equal] & \mathcal{T}(S_i, Y[-1]) \arrow[r] & \mathcal{T}(S_i, S_i^r) \arrow[r, "\mathcal{T}(S_i, g)"] \arrow[d, "\cong"] & \mathcal{T}(S_i, X) \arrow[r] \arrow[d, "\cong"] & \mathcal{T}(S_i, Y) \arrow[r] \arrow[d, equal] & \mathcal{T}(S_i, S_i^r[1]) \arrow[d, equal] \\
0 & & k^r & k^r & 0 & 0
\end{tikzcd}$$

The first term above is zero since it is isomorphic to $\mathcal{T}(X, S_i) = 0$, the last term is zero since $\mathcal{T}(S_i, S_i[1]) = 0$. Furthermore, $\mathcal{T}(S_i, g)$ must be an isomorphism since it is an epimorphism by definition, while its domain and co-domain have the same k-dimension. It follows that $\mathcal{T}(Y, S_i) \cong \mathcal{T}(S_i, Y[-1]) = 0$. Now Y satisfies $\mathcal{T}(Y, S'_j) = 0$ for all j, and hence as we argued above for X, we must have $\mathcal{T}(S_i, Y) \neq 0$ or else $Y = 0$. We have already seen that the first possibility does not occur, so suppose $Y = 0$. Then, by definition of Y, $X \cong S_1^r$, which contradicts $\mathcal{T}(X, S_1) = 0$.

Finally, for condition (4), observe that $\mathcal{T}(S_i[2], S_i) \cong \mathcal{T}(S_i, S_i[1]) = 0$, which implies that $S_i[2] \ncong S_i$. If $j \neq i$ and $n_j > 0$, then the exact sequence $\mathcal{T}(S'_j[2], S_i) \to \mathcal{T}(S_i^{n_j}[1], S_i) \to \mathcal{T}(S_j, S_i)$, obtained from applying $\mathcal{T}(-, S_i)$ to the triangle defining S'_j, yields $\mathcal{T}(S'_j[2], S_i) \neq 0$, as $\mathcal{T}(S_j, S_i) = 0$ while $\mathcal{T}(S_i^{n_j}[1], S_i) \cong$

$\mathcal{T}(S_i, S_i^{n_j}) \neq 0$. Thus $S_j'[2] \ncong S_j'$. Alternatively, if $n_j = 0$, then $S_j' = S_j[-1]$ and $S_j'[2] \ncong S_j'$ follows from $S_j[2] \ncong S_j$. $\square$

Remark. Recently, Koenig and Liu have defined *simple-minded systems* of objects (in stable categories) [12], and the definition carries over easily to a triangulated category $\mathcal{T}$ as above. Furthermore, they note that any simple minded system in this setting is also a maximal system of orthogonal bricks; although the converse is not clear. It turns out that the set of simple-minded systems in $\mathcal{T}$ is also closed under the mutations defined above, but the argument requires a closer study of torsion pairs in $\mathcal{T}$ and hence will be presented in a sequel to this article [7].

We can also define left mutations on the sets of maximal systems of orthogonal bricks in $\mathcal{T}$. Suppose $\mathcal{S} = \{S_1, \dots, S_n\}$ is a maximal system of orthogonal bricks in $\mathcal{T}$ with $\mathcal{T}(S_i, S_i[1]) = 0$. Then the **left mutation of** $\mathcal{S}$ **at** S_i is defined as the set $\mu_i^-(\mathcal{S}) = \{S_1', \dots, S_n'\}$ where $S_i' = S_i$ and for $j \neq i$, S_j' is defined via a distinguished triangle

$$S_i^{n_j} \xrightarrow{g_j} S_j[1] \to S_j' \to$$

where g_j is a minimal right $\mathrm{add}(S_i)$-approximation. As expected, left and right mutation at S_i turn out to be inverse operations. To see this, it suffices to observe that in the triangle defining $S_j' \in \mu_i^+(\mathcal{S})$, the connecting morphism $S_i^{n_j} \to S_j'[1]$ is a right $\mathrm{add}(S_i)$-approximation (since $\mathcal{T}(S_i, S_j) = 0$). In particular, $S_j'' \in \mu_i^-(\mu_i^+(\mathcal{S}))$ is defined as the cone of this connecting morphism, which is just S_j. Hence we have the following.

Proposition 6.3. *For any maximal system of orthogonal bricks $\mathcal{S}$ in $\mathcal{T}$ such that $\mathcal{T}(S_i, S_i[1]) = 0$, we have $\mu_i^+(\mu_i^-(\mathcal{S})) = \mathcal{S} = \mu_i^-(\mu_i^+(\mathcal{S}))$.*

We now explain how these mutations keep track of the images of the simple modules under the stable equivalences induced by successive tilting mutations. More specifically, suppose that $F : D^b(A') \to D^b(A)$ is an equivalence of triangulated categories, inducing an equivalence $\underline{F} : \underline{\mathrm{mod}}\text{-}A' \to \underline{\mathrm{mod}}\text{-}A$, and $\mathcal{S} = \{\underline{F}(S_1'), \dots, \underline{F}(S_n')\}$ where the S_i' are the simple A'-modules. Setting $A'' = \mu_i^+(A')$, we get a stable equivalence $\underline{G} : \underline{\mathrm{mod}}\text{-}A'' \to \underline{\mathrm{mod}}\text{-}A'$, and assuming $\mathrm{Ext}^1_{A'}(S_i', S_i') = 0$, Theorem 4.1 implies that the image of the set of simple A''-modules $\mathcal{S}'' = \{S_1'', \dots, S_n''\}$ under $\underline{G}$ is $\mu_i^+(S_1', \dots, S_n')$. Thus $\underline{F}(\underline{G}(\mathcal{S}'')) = \underline{F}(\mu_i^+(S_1', \dots, S_n')) = \mu_i^+(\mathcal{S})$, since $\underline{F}$ preserves triangles and approximations. In fact, by general arguments involving the preservation of exact sequences by a stable equivalence (see for example [14]), the same is true if $\underline{F}$ is replaced by any stable equivalence between algebras without nodes, even one that is not assumed to be an equivalence of triangulated categories.

Corollary 6.4. *Let A be a weakly symmetric algebra, and let $\mathcal{M}_{st}$ (respectively, $\mathcal{M}_{der}$) be the set of all maximal systems of orthogonal bricks in $\underline{\mathrm{mod}}$-A which correspond to the simple B-modules via a stable equivalence $\underline{\mathrm{mod}}\text{-}B \to \underline{\mathrm{mod}}\text{-}A$ (respectively, via a stable equivalence induced by a derived equivalence $\mathcal{D}^b(B) \to D^b(A)$) for some algebra B. Then $\mathcal{M}_{st}$ (resp. $\mathcal{M}_{der}$) is closed under left and right mutation.*

References

[1] T. Aihara. *Mutating Brauer trees.* Preprint (2011) arXiv:1009.3210v3.

[2] H. Asashiba. *The derived equivalence classification of representation-finite selfinjective algebras.* J. Algebra 214 (1999), no. 1, 182–221.

[3] H. Asashiba. *On a lift of an individual stable equivalence to a standard derived equivalence for representation-finite self-injective algebras.* Algebr. Represent. Theory 6 (2003), no. 4, 427–447.

[4] J. Chuang and J. Rickard. *Representations of finite groups and tilting.* Handbook of tilting theory, 359–391, London Math. Soc. Lecture Note Ser., 332, Cambridge Univ. Press, Cambridge, 2007.

[5] H. Derksen, J. Weyman and A. Zelevinsky. *Quivers with potentials and their representations. I. Mutations.* Selecta Math. (N.S.) 14 (2008), no. 1, 59–119.

[6] A. Dugas. *A construction of derived equivalent pairs of symmetric algebras.* Preprint (2010) arXiv:1005.5152v2.

[7] A. Dugas. *Torsion pairs and simple minded systems in triangulated categories.* In preparation.

[8] T. Holm. *Derived equivalent tame blocks.* J. Algebra 194 (1997) 178–200.

[9] T. Holm. *Derived equivalence classification of algebras of dihedral, semidihedral, and quaternion type.* J. Algebra 211 (1999) 159–205.

[10] M. Hoshino and Y. Kato. *Tilting complexes defined by idempotents.* Comm. Algebra 30 (2002), no. 1, 83–100.

[11] W. Hu and C. Xi. *Derived equivalences and stable equivalences of Morita type, I.* Nagoya Math. J. 200 (2010) 107–152.
[12] S. Koenig and Y. Liu. *Simple-minded systems in stable module categories.* Preprint (2010) arXiv:1009.1427v1.
[13] M. Linckelmann. *Stable equivalences of Morita type for self-injective algebras and p-groups.* Math. Z. 223 (1996), no. 1, 87–100.
[14] R. Martínez Villa. *Properties that are left invariant under stable equivalence.* Comm. Algebra 18 (1990), no. 12, 4141–4169.
[15] T. Okuyama. *Some examples of derived equivalent blocks of finite groups.* Preprint (1998).
[16] Z. Pogorzały. *Algebras stably equivalent to self-injective special biserial algebras.* Comm. in Algebra 22 (1994), no. 4, 1127–1160.
[17] J. Rickard. *Morita theory for derived categories.* J. London Math. Soc. 39 (1989), no. 2, 436-456.
[18] J. Rickard. *Derived categories and stable equivalence.* J. Pure Appl. Algebra 61 (1989), no. 3, 303-317.
[19] J. Vitoria. *Mutations vs. Seiberg duality.* J. Algebra 321 (2009), no. 3, 816–828.

Department of Mathematics, University of the Pacific, 3601 Pacific Ave, Stockton CA 95211, USA
E-mail address: adugas@pacific.edu